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ZORICH CONJECTURE FOR HYPERELLIPTIC RAUZY–VEECH GROUPS

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ABSTRACT. We describe the structure of hyperelliptic Rauzy diagrams and hyperelliptic Rauzy–Veech groups. In particular, this provides a solution of the hyperelliptic cases of a conjecture of Zorich on the Zariski closure of Rauzy–Veech groups.

1. INTRODUCTION

The Kontsevich–Zorich conjecture provides a precise description of the deviations of ergodic averages of almost every interval exchange transformations and translation flows in terms of the Lyapunov exponents of the Kontsevich–Zorich (KZ) cocycle with respect to the Masur–Veech measures on the strata of moduli spaces of translation surfaces.

After an important partial progress of Forni [8] in 2001, the Kontsevich–Zorich conjecture was fully established by Avila and Viana [3] in 2007 via the study of certain combinatorial models for the Kontsevich–Zorich cocycles called Rauzy–Veech groups. In a nutshell, Avila and Viana confirmed the Kontsevich–Zorich conjecture by showing that the Rauzy–Veech groups are pinching and twisting.

Nevertheless, Avila and Viana pointed out in [3, Remark 6.12] that their methods leave open an interesting conjecture of Zorich (cf. [14, Appendix A.3]) concerning the Zariski denseness of Rauzy–Veech groups in symplectic groups. Indeed, it is known[1] among experts that some pinching and twisting groups have small Zariski closures, so that it is not possible to abstractly deduce[2] Zorich's conjecture from Avila–Viana techniques.

In this paper, we confirm Zorich conjecture for *hyperelliptic* Rauzy–Veech groups by proving the following stronger result.

Theorem 1.1. *The Rauzy–Veech group associated to a hyperelliptic connected component of a stratum of the moduli space of genus g translation surfaces is an explicit, finite-index subgroup of the symplectic group $Sp(2g, \mathbb{Z})$.*

We refer the reader to Theorem 2.9 below for a precise version of this statement. For now, let us just make some comments on the proof of this result.

Rauzy [11] discovered a particularly beautiful combinatorial description for hyperelliptic Rauzy diagrams. This description allows us to compute the generators of hyperelliptic Rauzy–Veech groups and, more importantly, to relate distinct hyperelliptic Rauzy–Veech groups via an inductive procedure. In particular, we are able to prove Theorem 1.1 by induction (on the complexity of the hyperelliptic Rauzy diagrams): see Section 3 below.

After we completed the argument in the above paragraph, Möller pointed out (in private communication) that our description of hyperelliptic Rauzy–Veech groups shared some

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[1]See the Appendix A below for a concrete example.

[2]On the other hand, Zorich conjecture implies Avila–Viana theorem on the pinching and twisting properties for Rauzy–Veech groups. In fact, Zariski density implies the pinching property by the work of Benoist [4], while the twisting property is automatic (because it has to do with minors of matrices). Hence, our proofs of Theorem 1.1 give new proofs of Avila–Viana theorem in the particular case of hyperelliptic Rauzy–Veech groups.

similarities with the work [1] of A'Campo on certain representations of braid groups defined via homological actions on hyperelliptic Riemann surfaces. As it turns out, this is not a coincidence: we show in Section 4 below that the hyperelliptic Rauzy–Veech groups are naturally related to the images of the monodromy representations considered by A'Campo. In particular, the main results of A'Campo's paper [1] can be used to give another proof of Theorem 1.1.

Remark 1.2. This second proof of Theorem 1.1 described in the previous paragraph provides more information about hyperelliptic Rauzy diagrams: for instance, we will show that the image of the natural homomorphism from the fundamental group of hyperelliptic Rauzy diagrams to the mapping class group is an infinite-index subgroup called symmetric mapping class group. In particular, the analog of Theorem 1.1 at the fundamental group level is not true. See Section 4 for more details.

The organization of this paper is the following. In Section 2, we recall some basic facts about hyperelliptic Rauzy diagrams and Rauzy–Veech groups, and we state in Theorem 2.9 the precise version of Theorem 1.1. In Section 3, we give our first proof of Theorem 2.9 by induction on the complexity of hyperelliptic Rauzy diagrams. In Section 4, we give a second proof of Theorem 2.9 based on the interpretation of hyperelliptic Rauzy–Veech groups in terms of certain monodromy representations of braid groups. In particular, Sections 3 and 4 can be read independently of each other. Finally, we exhibit in Appendix A an example of pinching and twisting group with small Zariski closure in order to justify our assertion that Zorich conjecture can not be abstractly reduced to the results of Avila–Viana [3].

Remark 1.3. In a forthcoming paper [2], we will use the framework of this article to analyze the Kontsevich-Zorich cocycle over certain loci of cyclic covers of hyperelliptic connected components of strata of the moduli space of translation surfaces.

Remark 1.4. In a recent preprint [5], Eskin, Filip and Wright studied the algebraic hull of the Kontsevich–Zorich cocycle and the monodromies associated to general ergodic $SL(2,\mathbb{R})$-invariant probability measures on moduli spaces of translation surfaces. The notion of Rauzy–Veech groups shares some similarities with the algebraic hulls and the monodromies of Masur–Veech measures: roughly speaking, Rauzy–Veech groups, resp. algebraic hulls, resp. monodromies, are related to matrices obtained by following certain orbits of the Teichmüller flow, resp. orbits of $SL(2,\mathbb{R})$, resp. arbitrary paths in connected components of strata of moduli spaces of translation surfaces. In particular, one has that Rauzy–Veech groups are subgroups of the monodromies of Masur–Veech measures. Consequently, our Theorem 1.1 implies that monodromies of hyperelliptic Masur–Veech measures are commensurable to arithmetic lattices of symplectic groups: this refines Corollary 1.7 in Filip's article [7] in this particular setting. On the other hand, the relation between Rauzy–Veech groups and algebraic hull of Masur–Veech measures is not so obvious (partly because the definition of algebraic hull involves representing matrices in *a priori* unknown measurably chosen bases) and, thus, it is not clear that our Theorem 1.1 provides any new information related to Corollary 1.4 in Eskin–Filip–Wright paper [5].

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2. THE HYPERELLIPTIC RAUZY–VEECH GROUP

In this entire section, we will assume that the reader has some familiarity with the lecture notes [13] by the third author of this paper. Also, let us point out that the facts stated in the next subsection are just reformulations (in our notations) of the results obtained by Rauzy [11, Section 4].

2.1. **Hyperelliptic Rauzy diagrams: notations and description.** Let $d \geqslant 2$ be an integer. Let $\mathcal{A}_d$ be the alphabet whose d elements are the integers in arithmetic progression $d-1,\, d-3, \dots, 1-d$. Let ι be the involution $k \mapsto -k$ of $\mathcal{A}_d$. We define inductively the hyperelliptic Rauzy class $\mathcal{R}_d$ over $\mathcal{A}_d$ and the associated hyperelliptic Rauzy diagram $\mathcal{D}_d$. The Rauzy class $\mathcal{R}_d$ contains a central vertex $\pi^* = \pi^*(d) = (\pi_t^*(d), \pi_b^*(d))$ associated to the pair of bijections $\pi_t^*(d) : \mathcal{A}_d \to \{1, \dots, d\}$ and $\pi_b^*(d) : \mathcal{A}_d \to \{1, \dots, d\}$ defined by

$$\pi_t^*(d)(k) = \frac{1}{2}(d+1+k), \quad \pi_b^*(d)(k) = \frac{1}{2}(d+1-k).$$

For $d = 2$, this is the only vertex. For $d \geqslant 2$, $\mathcal{R}_{d+1}$ is the disjoint union of $\pi^*(d+1)$, $j_t(\mathcal{R}_d)$ and $j_b(\mathcal{R}_d)$, where the injective maps j_t, j_b are defined as follows: for $\pi \in \mathcal{R}_d$, writing $j_t(\pi) = t\pi$, $j_b(\pi) = b\pi$, we have that $t\pi = (t\pi_t, t\pi_b)$ and $b\pi = (b\pi_t, b\pi_b)$ are given by the bijections from $\mathcal{A}_d$ to $\{1, \dots, d\}$ described by the formulas

$$t\pi_t(-d) = 1, \qquad t\pi_b(-d) = \pi_b(d-3),$$

$$t\pi_t(k) = 1 + \pi_t(k-1),$$

$$t\pi_b(k) = \begin{cases} \pi_b(k-1) & \text{if } \pi_b(k-1) < \pi_b(d-3), \\ \pi_b(k-1)+1 & \text{if } \pi_b(k-1) \geqslant \pi_b(d-3), \end{cases}$$

for $2-d \leqslant k \leqslant d$, and

$$b\pi_b(d) = 1, \qquad b\pi_t(d) = \pi_t(3-d),$$

$$b\pi_b(k) = 1 + \pi_b(k+1),$$

$$b\pi_t(k) = \begin{cases} \pi_t(k+1) & \text{if } \pi_t(k+1) < \pi_t(3-d), \\ \pi_t(k+1)+1 & \text{if } \pi_b(k+1) \geqslant \pi_t(3-d), \end{cases}$$

for $-d \leqslant k \leqslant d-2$.

The one-to-one maps R_t, resp. R_b from $\mathcal{R}_d$ to itself determining the arrows of $\mathcal{D}_d$ of top, resp. bottom type verify

$$\begin{cases} R_t(\pi^*(d+1)) = j_t(\pi^*(d)), \\ R_b(\pi^*(d+1)) = j_b(\pi^*(d)), \end{cases} \qquad \begin{cases} R_t \circ j_b \circ R_t^{-1} = j_b, \\ R_b \circ j_t \circ R_b^{-1} = j_t, \end{cases}$$

$$\begin{cases} R_t \circ j_t \circ R_t^{-1}(\pi) = j_t(\pi), & \pi \neq \pi^*(d), \\ R_b \circ j_b \circ R_b^{-1}(\pi) = j_b(\pi), & \pi \neq \pi^*(d), \end{cases}$$

$$R_t \circ j_t \circ R_t^{-1}(\pi^*(d)) = \pi^*(d+1) = R_b \circ j_b \circ R_b^{-1}(\pi^*(d)).$$

The involution I_d on $\mathcal{R}_d$ defined by $I_d((\pi_t, \pi_b)) := (\pi_b \circ \iota, \pi_t \circ \iota)$ satisfies

$$I_d(\pi^*(d)) = \pi^*(d), \quad I_{d+1} \circ j_b \circ I_d = j_t, \quad I_d \circ R_b \circ I_d = R_t.$$

There is a natural one-to-one correspondence W_d between the elements of $\mathcal{R}_d$ and the words in $\{t, b\}$ of length $< d-1$: namely, $W_d(\pi^*(d))$ is the empty word, $W_d(j_t(\pi))$ is the word $tW_{d-1}(\pi)$ and $W_d(j_b(\pi))$ is the word $bW_{d-1}(\pi)$. The involution I_d corresponds to the exchange of the letters t, b. One has also

$$W_d(R_t(\pi)) = W_d(\pi)t, \quad W_d(R_b(\pi)) = W_d(\pi)b, \quad \text{if } |W_d(\pi)| < d-2.$$

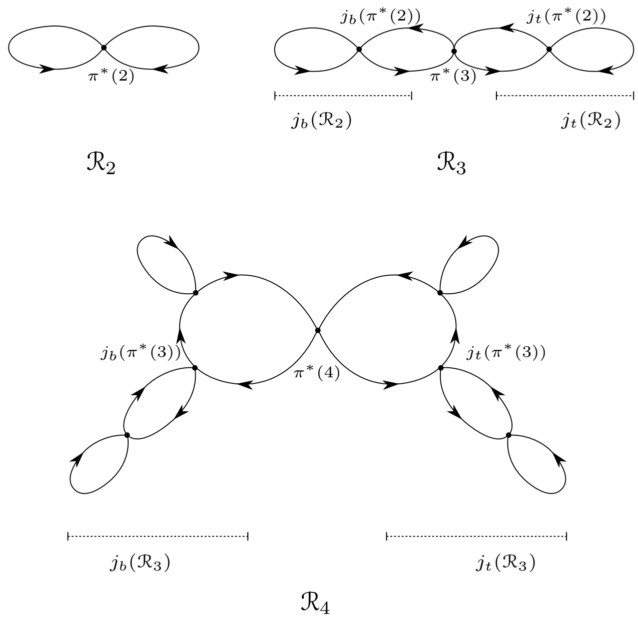

FIGURE 1. Geometry of the hyperelliptic Rauzy classes $\mathcal{R}_2$, $\mathcal{R}_3$ and $\mathcal{R}_4$.

When $|W_d(\pi)| = d-2$, one writes $W_d(\pi) = W't^m$ with $m \geqslant 0$ and W' empty or finishing by b; one has then $W_d(R_t(\pi)) = W'$. Similarly for $W_d(R_b(\pi))$.

It is also not difficult to recover from $W_d(\pi)$ the winners of the arrows starting from π: the winner of the arrow of top type starting from π is the letter $d-1-2w_b(\pi)$ of $\mathcal{A}_d$, where $w_b(\pi)$ is the number of occurrences of b in $W_d(\pi)$; similarly, the winner of the arrow of bottom type starting from π is the letter $1-d+2w_t(\pi)$ of $\mathcal{A}_d$. Observe that we have always

$$d-1-2w_b(\pi) > 1-d+2w_t(\pi).$$

Another useful property of the hyperelliptic Rauzy diagrams is the following: given any vertex $\pi \in \mathcal{R}_d$, there is an *unique* oriented *simple*[3] path in $\mathcal{D}_d$ from $\pi^*(d)$ to π. Indeed, this is best seen via the correspondence W_d above: the length of such a path is $|W_d(\pi)|$ and the path itself is through the sequence of initial subwords of $W_d(\pi)$. We will denote by $\gamma^*(\pi)$ this path.

Observe that all simple loops of positive length in $\mathcal{R}_d$ are *elementary*, that is, they are made of arrows of the same type (and consequently with the same winner). For any such loop γ, there is a unique vertex π such that γ passes through π but $\gamma^*(\pi)$ does not contain any arrow of γ. As it turns out, π is the vertex of γ such that $|W_d(\pi)|$ is minimal. One has

$$|\gamma| + |W_d(\pi)| = d-1.$$

2.2. **The hyperelliptic Rauzy–Veech group.** Let γ be an elementary simple loop in $\mathcal{R}_d$ and denote by π the vertex of γ with $|W_d(\pi)|$ minimal (see above). Let γ' be the non-oriented loop based at $\pi^*(d)$ defined by $\gamma' = \gamma^*(\pi) * \gamma * (\gamma^*(\pi))^{-1}$.

[3] A path is *simple* if it does not pass more than once through any vertex.

2.2.1. *Calculation of some Kontsevich–Zorich matrices.* Let us compute the matrix $B_{\gamma'}$ associated to γ' by the Rauzy–Veech algorithm / Kontsevich–Zorich cocycle (see Subsection 7.5 in [13] for definitions). Assume for instance that the loop γ is of top type. We have $w_b(\pi) + w_t(\pi) + |\gamma| = d - 1$.

On one hand, the winner of all the arrows of γ is $d - 1 - 2w_b(\pi)$. On the other hand, starting from π, the losers are successively $1 - d + 2w_t(\pi), 3 - d + 2w_t(\pi), \dots, d - 3 - 2w_b(\pi)$. Therefore, by writing $B_\gamma(v) = v'$, we have

$$v'_k = \begin{cases} v_k + v_{d-1-2w_b(\pi)} & \text{if } 1 - d + 2w_t(\pi) \leqslant k < d - 1 - 2w_b(\pi), \\ v_k & \text{otherwise} \end{cases}$$

If $W_d(\pi)$ is empty, i.e., $w_b(\pi) = w_t(\pi) = 0$, we have $\gamma' = \gamma$ and $B_{\gamma'} = B_\gamma$. So, we can assume now that $W_d(\pi)$ is not empty. Let w_1 be the number of occurrences of b at the end of $W_d(\pi)$; one has $0 < w_1 \leqslant w_b(\pi)$. Write $\gamma^*(\pi) = \gamma^1 * \gamma_1$ with $|\gamma_1| = w_1$.

The winner of all the arrows of γ_1 is $1 - d + 2w_t(\pi)$, while the losers are successively $d - 1 - 2w_b(\pi) + 2w_1, \dots, d - 1 - 2w_b(\pi) + 2w_1$. Thus, by writing $B_{\gamma_1}(v) = \widehat{v}$, we have

$$\widehat{v}_k = \begin{cases} v_k + v_{1-d+2w_t(\pi)} & \text{if } d - 1 - 2w_b(\pi) < k \leqslant d - 1 - 2w_b(\pi) + 2w_1, \\ v_k & \text{otherwise} \end{cases}$$

For $B_{\gamma_1 * \gamma * \gamma_1^{-1}}(v) = v'$, we have therefore

$$v'_k = \begin{cases} v_k + v_{d-1-2w_b(\pi)} & \text{if } 1 - d + 2w_t(\pi) \leqslant k < d - 1 - 2w_b(\pi), \\ v_k - v_{d-1-2w_b(\pi)} & \text{if } d - 1 - 2w_b(\pi) < k \leqslant d - 1 - 2w_b(\pi) + 2w_1, \\ v_k & \text{otherwise} \end{cases}$$

Let $\gamma'_1 := \gamma_1 * \gamma * \gamma_1^{-1}$. If γ^1 is empty, i.e., $w_b(\pi) = w_1$, $w_t(\pi) = 0$, we have $\gamma' = \gamma'_1$ and the computation of $B_{\gamma'}$ is complete. Otherwise, we go on by writing $\gamma^1 = \gamma^2 * \gamma_2$, with γ_2 made of the arrows of top type ending γ^1. By writing $|\gamma_2| = w_2 > 0$ and $B_{\gamma_2}(v) = \widehat{v}$, one has

$$\widehat{v}_k = \begin{cases} v_k + v_{d-1-2w_b(\pi)+2w_1} & \text{if } 1 - d + 2w_t(\pi) - 2w_2 \leqslant k < 1 - d + 2w_t(\pi), \\ v_k & \text{otherwise} \end{cases}$$

Let $\gamma'_2 = \gamma_2 * \gamma'_1 * \gamma_2^{-1}$. For $B_{\gamma'_2}(v) = v'$, we have

$$v'_k = \begin{cases} v_k + v_{d-1-2w_b(\pi)} & \text{if } 1 - d + 2w_t(\pi) - 2w_2 \leqslant k < d - 1 - 2w_b(\pi), \\ v_k - v_{d-1-2w_b(\pi)} & \text{if } d - 1 - 2w_b(\pi) < k \leqslant d - 1 - 2w_b(\pi) + 2w_1, \\ v_k & \text{otherwise} \end{cases}$$

We go on till γ^m is empty. In this way, for $B_{\gamma'}(v) = v'$, one obtains:

$$v'_k = \begin{cases} v_k + v_{d-1-2w_b(\pi)} & \text{if } k < d - 1 - 2w_b(\pi), \\ v_k - v_{d-1-2w_b(\pi)} & \text{if } k > d - 1 - 2w_b(\pi), \\ v_k & \text{if } k = d - 1 - 2w_b(\pi). \end{cases}$$

Note that this formula depends only on the type and the winner of γ.

Similarly, when γ has bottom type, the formula for $B_{\gamma'}(v) = v'$ is

$$v'_k = \begin{cases} v_k + v_{1-d+2w_t(\pi)} & \text{if } k > 1 - d + 2w_t(\pi), \\ v_k - v_{1-d+2w_t(\pi)} & \text{if } k < 1 - d + 2w_t(\pi), \\ v_k & \text{if } k = 1 - d + 2w_t(\pi). \end{cases}$$

Remark 2.1. The matrix $B_{\gamma'}$ associated to a loop γ of bottom type is the inverse of the matrix corresponding to the loop of top type and the same winner as γ.

Remark 2.2. Actually, one can completely describe the action of γ' at the homotopy level (instead of the homology level of the matrix $B_{\gamma'}$) and, again, it depends only on the type and winner of γ. We will come back to this point later in Section 4 below.

2.2.2. *Definition of the hyperelliptic Rauzy–Veech groups.* Let γ be the elementary simple loop in $\mathcal{R}_d$ of bottom type with winner $p \in \mathcal{A}_d$. Our previous discussion shows that the matrix $B_{\gamma'}$ corresponds to the operator L_p on $\mathbb{Z}^{\mathcal{A}_d}$ given by

$$L_p(e_q) = \begin{cases} e_q & \text{if } q \neq p \\ -\sum_{r<p} e_r + \sum_{r\geqslant p} e_r & \text{if } q = p \end{cases}$$

where (e_p) is the canonical basis of $\mathbb{C}^{\mathcal{A}_d}$. Also, by Remark 2.1, the matrix associated to the elementary simple loop in $\mathcal{R}_d$ of top type with winner $p \in \mathcal{A}_d$ corresponds to the inverse of the operator L_p.

Definition 2.3. The *hyperelliptic Rauzy–Veech group of complexity* d is the subgroup $\mathfrak{G}_d$ of $SL(\mathbb{Z}^{\mathcal{A}_d})$ generated by the operators L_p, $p \in \mathcal{A}_d$.

2.2.3. *Intersection form.* The antisymmetric matrix $\Omega = \Omega(d)$ with entries

$$\Omega_{pq} := \begin{cases} +1 & \text{if} \quad p < q \\ -1 & \text{if} \quad p > q \\ 0 & \text{if} \quad p = q \end{cases}$$

indexed by $\mathcal{A}_d \times \mathcal{A}_d$ can be interpreted as the intersection form on the homology of certain translation surfaces (see Subsections 3.4 and 4.5 of [13]).

The operators L_p satisfy $L_p \, \Omega \, {}^tL_p = \Omega$ and, *a fortiori*, the same is true of all the elements of $\mathfrak{G}_d$:

$$B \, \Omega \, {}^tB = \Omega, \qquad \forall B \in \mathfrak{G}_d. \tag{2.1}$$

2.2.4. *Symplecticity for d even.* The matrix $\Omega(d)$ corresponds to the intersection form on the absolute homology of certain translation surfaces when d is even. In particular, $\Omega(d)$ is unimodular. The symplectic form on $\mathbb{Z}^{\mathcal{A}_d}$ is defined by

$$(v, v') \mapsto {}^t v \, \Omega^{-1} v'.$$

The relation (2.1) shows that $\mathfrak{G}_d \subset Sp(\Omega^{-1}(d), \mathbb{Z})$. Note that the group $Sp(\Omega^{-1}(d), \mathbb{Z})$ is isomorphic to $Sp(d, \mathbb{Z})$.

2.2.5. *The case of d odd.* Assume that d is odd.

Lemma 2.4. *The matrix $\Omega(d)$ has rank $d-1$. The image $\Omega(d)(\mathbb{Z}^{\mathcal{A}_d})$ is the hyperplane*

$$H(d) = \left\{ v \in \mathbb{Z}^{\mathcal{A}_d} \;\middle|\; \sum_{p \in \mathcal{A}_d} (-1)^{p/2} v_p = 0 \right\}.$$

The kernel of $\Omega(d)$ is generated by $h^\star := \sum_{p\in\mathcal{A}_d} (-1)^{p/2} e_p$

Proof. The vectors $\Omega(d)e_p$ belong to $H(d)$. Moreover, one has

$$\Omega(d)(e_{p+2} - e_p) = e_p + e_{p+2}, \qquad \forall \, p \in \mathcal{A}_d, \, p < d-1,$$

and $\{e_p + e_{p+2} : p \in \mathcal{A}_d, \, p < d-1\}$ form a basis of $H(d)$. Finally, it is clear that $\Omega(d).h^\star = 0$. □

Proposition 2.5. *The matrices in $\mathfrak{G}_d$ satisfy ${}^t h^\star B = {}^t h^\star$*

Proof. Indeed this is the case for each L_p, $p \in \mathcal{A}_d$. □

The symplectic form induced by $\Omega(d)$ on $H(d)$ is defined as follows: for $v, v' \in H(d)$, $v = \Omega w, v' = \Omega w'$, we set

$$(v, v') \mapsto {}^t w\, \Omega\, w'.$$

Observe that this does not depend on the choices of w, w'.

Remark 2.6. This is coherent with the definition of the symplectic form for d even.

From (2.1) (or Proposition 2.5), the elements of $\mathfrak{G}_d$ preserve the hyperplane $H(d)$ and their restrictions to $H(d)$ are symplectic with respect to the symplectic form on $H(d)$.

We denote still by $Sp(\Omega^{-1}(d), \mathbb{Z})$ the group of operators in $SL(\mathbb{Z}^{\mathcal{A}_d})$ satisfying (2.1) (although Ω is not invertible in this case).

2.2.6. *Reduction modulo* 2. For $p \in \mathcal{A}_d$, let $\bar{L}_p$ be the reduction mod.2 of L_p: it acts on $(\mathbb{Z}/2)^{\mathcal{A}_d}$. Denote by $(\bar{e}_p)_{p \in \mathcal{A}_d}$ the canonical basis of $(\mathbb{Z}/2)^{\mathcal{A}_d}$ and define $\bar{e}^\star := \sum_{p \in \mathcal{A}_d} \bar{e}_p$.

Proposition 2.7. *For any $q \in \mathcal{A}_d$, the $d+1$ vectors $e^\star$, e_p, $p \in \mathcal{A}_d$, are permuted by $\bar{L}_q$. More precisely, $\bar{L}_q$ fixes $\bar{e}_p$ for $p \neq q$ and exchanges $\bar{e}_q$ and $\bar{e}^\star$.*

Proof. This follows easily from the definitions. □

Corollary 2.8. *The image of $\mathfrak{G}_d$ in $SL(\mathbb{F}_2^{\mathcal{A}_d})$ is the subgroup $\mathfrak{H}_d$ formed of elements preserving $\mathcal{E} := \{\bar{e}^\star\} \cup \{\bar{e}_p\}_{p \in \mathcal{A}_d}$. It is isomorphic to the symmetric group of order $d+1$.*

Proof. Indeed, this is a direct consequence of Proposition 2.7 because the group generated by the transpositions $(0, i)$, $1 \leqslant i \leqslant d$ is the full symmetric group of $\{0, \dots, d\}$. □

2.3. **Statement of the main result.** The following statement provides a precise version for Theorem 1.1 above.

Theorem 2.9. *For any integer $d \geqslant 2$, the group $\mathfrak{G}_d$ consists of matrices $B \in Sp(\Omega^{-1}(d), \mathbb{Z})$ whose image in $SL(\mathbb{F}_2^{\mathcal{A}_d})$ belongs to $\mathfrak{H}_d$.*

Here, we recall that the group $Sp(\Omega^{-1}(d), \mathbb{Z})$ was defined using a special convention when d is odd (see the two paragraphs after Remark 2.6 above).

In the sequel, we will give two proofs of this result in Sections 3 and 4.

More precisely, our discussion in Section 3 below will establish (by induction) this theorem at the same time of the next two results.

Theorem 2.10. *For any even integer $d \geqslant 2$ and any $p \in \mathcal{A}_d$, the orbit of e_p under $\mathfrak{G}_d$ is equal to the set of primitive vectors in $\mathbb{Z}^{\mathcal{A}_d}$ which are congruent mod.2 to a vector in the set $\mathcal{E}$ from Corollary 2.8.*

Theorem 2.11. *For any odd integer $d \geqslant 3$, any $p \in \mathcal{A}_d$, the orbit of e_p under $\mathfrak{G}_d$ is equal to the set of primitive vectors in $\mathbb{Z}^{\mathcal{A}_d}$ which are congruent mod.2 to a vector in $\mathcal{E}$ and belong to the affine hyperplane*

$$\{v \in \mathbb{Z}^{\mathcal{A}_d} | {}^t h^\star.(v - e_p) = 0\}.$$

On the other hand, our discussion in Section 4 below will establish Theorem 2.9 by expanding on Remark 2.2 above, that is, we will use the relationship between hyperelliptic Rauzy–Veech groups and certain monodromy representations of braid groups in order to reduce Theorem 2.9 to some results of A'Campo [1].

3. Proof of Theorem 2.9

In this section, we prove Theorems 2.9, 2.10 and 2.11 by induction on the integer $d \geqslant 2$. In the initial case $d = 2$, it is well known that the group generated by L_{-1} and L_1 is equal to $SL(\mathbb{Z}^{\mathcal{A}_2})$. Observe that $\mathfrak{H}_2$ is equal to $SL(\mathbb{F}_2^{\mathcal{A}_2})$ and $Sp(\Omega^{-1}(2), \mathbb{Z})$ is equal to $SL(\mathbb{Z}^{\mathcal{A}_2})$. Therefore Theorem 2.9 holds for $d = 2$. Any primitive vector in $\mathbb{Z}^{\mathcal{A}_2}$ belongs to the orbit of e_1 (or e_{-1}) under $SL(\mathbb{Z}^{\mathcal{A}_2})$. Therefore Theorem 2.10 also holds for $d = 2$.

In the sequel, we denote by $\mathfrak{G}'_d$ the group of matrices $B \in Sp(\Omega^{-1}(d), \mathbb{Z})$ whose image in $SL(\mathbb{F}_2^{\mathcal{A}_d})$ belongs to $\mathfrak{H}_d$. By Proposition 2.7 and relation (2.1), the group $\mathfrak{G}_d$ is contained in $\mathfrak{G}'_d$. In this setting, our task of showing Theorem 2.9 consists in proving that $\mathfrak{G}_d$ is equal to $\mathfrak{G}'_d$.

3.1. **Stabilizer of e_{1-d} in $\mathfrak{G}'_d$.** A matrix B belonging to the stabilizer $\mathfrak{K}_d$ of e_{1-d} in $\mathfrak{G}'_d$ can be written in the block form

$$B := \begin{pmatrix} 1 & v \\ 0 & g \end{pmatrix}. \tag{3.1}$$

Here, v is a integral line vector of dimension $d-1$ and g is an unimodular square matrix of dimension $d-1$. Both are indexed by $\mathcal{A}_d \setminus \{1-d\}$, which is equal to $\mathcal{A}_{d-1}$ shifted by 1. When considering the stabilizer $\mathfrak{K}_d$, we will forget the shift and think of v, g as indexed by $\mathcal{A}_{d-1}$.

Let $e^\sharp := \sum_{p \in \mathcal{A}_d \setminus \{1-d\}} e_p$. By writing

$$\Omega(d) = \begin{pmatrix} 0 & {}^t e^\sharp \\ -e^\sharp & \Omega(d-1) \end{pmatrix},$$

the relation (2.1) is equivalent to

$$\begin{cases} g\,\Omega(d-1)\,{}^t g &= \Omega(d-1) \\ \Omega(d-1)\,{}^t v &= e^\sharp - g^{-1} e^\sharp \end{cases} \tag{3.2}$$

Here, the first relation means that $g \in Sp(\Omega^{-1}(d-1), \mathbb{Z})$. The map $B \mapsto g$ defines a homomorphism φ_d from $\mathfrak{K}_d$ to $Sp(\Omega^{-1}(d-1), \mathbb{Z})$.

Proposition 3.1. *The image of this homomorphism is equal to $\mathfrak{G}'_{d-1}$.*

Proof. First, the image is contained into $\mathfrak{G}'_{d-1}$: if B is congruent mod.2 to a matrix in $\mathfrak{H}_d$, g is congruent mod.2 to a matrix in $\mathfrak{H}_{d-1}$. For the converse, let $g \in \mathfrak{G}'_{d-1}$. We first observe that, when d is even, the vector $e^\sharp - g^{-1} e^\sharp$ is contained in the image $H(d-1)$ of $\Omega(d-1)$: indeed, one has (with $h^\star = \sum_{p \in \mathcal{A}_{d-1}} (-1)^{p/2} e_p$)

$${}^t h^\star (ge - e) = 0, \qquad \forall\, g \in Sp(\Omega^{-1}(d-1), \mathbb{Z}),\ \forall\, e \in \mathbb{Z}^{\mathcal{A}_{d-1}} \tag{3.3}$$

according to Proposition 2.5.

We now check that it is always possible to choose a solution v of the second equation in (3.2) such that B is congruent mod.2 to a matrix in $\mathfrak{H}_d$. There are two cases:

- The reduction mod.2 of g permutes the $\bar{e}_p$, $p \in \mathcal{A}_{d-1}$. In this case, the vector $e^\sharp - g^{-1} e^\sharp$ is even and one can find an even vector v which satisfies the second equation of (3.2). Then the reduction mod.2 of B belongs to $\mathfrak{H}_d$.
- There exists $p \in \mathcal{A}_{d-1}$ such that $g.e_p$ is congruent mod.2 to $e^\sharp$. Then $e^\sharp - g^{-1} e^\sharp$ is congruent mod.2 to $e^\sharp - e_p$, which is itself congruent mod.2 to $\Omega(d-1) e_p$. Therefore one can find a solution v of the second equation of (3.2) which is congruent mod.2 to ${}^t e_p$. Then the reduction mod.2 of B belongs to $\mathfrak{H}_d$.

This proves the proposition. □

Proposition 3.2. *When d is odd, the homomorphism φ_d is an isomorphism.*

Proof. Indeed, $\Omega(d-1)$ is invertible in this case, hence the second equation in (3.2) has an unique solution. □

Proposition 3.3. *When d is even, two matrices $B_0, B_1 \in \mathfrak{K}_d$ as in (3.1) have the same image under φ_d if and only if the difference $v_1 - v_0$ of the corresponding vectors is an even multiple of ${}^t h^\star$.*

Proof. Indeed $h^\star$ is a basis of the 1-dimensional kernel of $\Omega(d-1)$. The assertion of the proposition results from the end of the proof of Proposition 3.1. □

The stabilizer $\mathfrak{K}_d$ of e_{1-d} in $\mathfrak{G}'_d$ is completely described by Propositions 3.1, 3.2, 3.3.

3.2. **The subgroups $\mathfrak{S}_{p,q}$ of $\mathfrak{G}_d$.** Let $p < q$ be distinct elements of $\mathcal{A}_d$. We denote by $\mathfrak{S}_{p,q}$ the subgroup of $\mathfrak{G}_d$ generated by L_p and L_q.

Let $B \in \mathfrak{S}_{p,q}$. As the vectors e_r, $r \neq p, q$, are fixed by both L_p and L_q, we have $B.e_r = e_r$ for such r. We denote by $B^\sharp$ the 2×2 matrix

$$B^\sharp := \begin{pmatrix} B_{p,p} & B_{p,q} \\ B_{q,p} & B_{q,q} \end{pmatrix}.$$

Lemma 3.4. (1) *The map $B \mapsto B^\sharp$ is an isomorphism from $\mathfrak{S}_{p,q}$ onto $SL(2,\mathbb{Z})$.*
(2) *The other coefficients of B in the p-th and q-th columns are given by*

$$B_{r,p} = \begin{cases} -1 + B_{p,p} - B_{q,p} & \text{if} \quad r < p \\ -1 + B_{p,p} + B_{q,p} & \text{if} \quad p < r < q \\ 1 - B_{p,p} + B_{q,p} & \text{if} \quad r > q \end{cases}$$

$$B_{r,q} = \begin{cases} 1 + B_{p,q} - B_{q,q} & \text{if} \quad r < p \\ -1 + B_{p,q} + B_{q,q} & \text{if} \quad p < r < q \\ -1 - B_{p,q} + B_{q,q} & \text{if} \quad r > q \end{cases}$$

Proof. The case $d = 2$ (cf. the end of Subsection 2.3) shows that $\psi : B \mapsto B^\sharp$ is onto $SL(2,\mathbb{Z})$. Let W be a word in $L_p^{\pm 1}, L_q^{\pm 1}$, B the corresponding element of $\mathfrak{S}_{p,q}$. We show, by induction on the length of W, that B is determined by $\psi(B)$, with the formulas of the lemma. This is true when W is the empty word. When W has positive length, let w be the last letter of W, write $W = W'.w$ and let B' be the matrix associated to W'. If for instance $w = L_q^\eta$, $\eta \in \{\pm 1\}$, one has, for all $r \in \mathcal{A}_d$

$$\begin{cases} B_{r,p} &= B'_{r,p} \\ B_{r,q} &= B'_{r,q} - \eta B'_{r,p} + \eta\varepsilon \end{cases} \qquad \text{where} \quad \varepsilon := \begin{cases} -1 & \text{if} \quad r < p \\ 0 & \text{if} \quad r = p \\ -1 & \text{if} \quad p < r < q \\ 0 & \text{if} \quad r = q \\ 1 & \text{if} \quad r > q \end{cases}$$

It follows that the formulas of the lemma for the $B'_{r,p}, B'_{r,q}$ imply the same formulas for the $B_{r,p}, B_{r,q}$. One deals similarly with the case $w = L_p^\eta$. This proves the lemma. □

3.3. **The induction step in the odd case.** In this subsection, we assume that $d \geqslant 3$ is odd and that Theorems 2.9, 2.10 hold for $d-1$.

Proposition 3.5. *The stabilizer of e_{1-d} in $\mathfrak{G}_d$ is equal to $\mathfrak{K}_d$. It is generated by the L_p, $p \in \mathcal{A}_d$, $p \neq 1-d$.*

Proof. As $\mathfrak{G}_d \subset \mathfrak{G}'_d$, this stabilizer is contained in $\mathfrak{K}_d$. Conversely, let $B \in \mathfrak{K}_d$. Write B as in (3.1). As $\mathfrak{G}_{d-1} = \mathfrak{G}'_{d-1}$ by the induction hypothesis, there exists in the subgroup generated by the L_p, $p \in \mathcal{A}_d, p \neq 1-d$ a matrix in form (3.1) with the same image than B in $\mathfrak{G}_{d-1}$. This matrix has to be equal to B by Proposition 3.2. □

Proposition 3.6. *The orbit of e_{d-1} under $\mathfrak{G}_d$ is equal to the set $\mathcal{O}_d$ of primitive vectors in $\mathbb{Z}^{\mathcal{A}_d}$ which belong to the affine hyperplane $\{{}^t h^\star.(v - e_{d-1}) = 0\}$ and are congruent mod.2 to a vector in $\mathcal{E} = \{\sum_{p \in \mathcal{A}_d} \bar{e}_p\} \cup \{\bar{e}_p\}_{p \in \mathcal{A}_d}$.*

Proof. The set $\mathcal{O}_d$ contains e_{d-1} and satisfies $L_p(\mathcal{O}_d) \subset \mathcal{O}_d$ for all $p \in \mathcal{A}_d$. Hence, $\mathcal{O}_d$ contains the orbit of e_{d-1} under $\mathfrak{G}_d$.

Conversely, let $v = \sum_{p \in \mathcal{A}_d} v_p e_p$ be a vector in $\mathcal{O}_d$.

- Assume first that the vector $v' := \sum_{p \in \mathcal{A}_d \setminus \{1-d\}} v_p e_p$ is primitive. In particular, it is not even, hence it is congruent mod.2 to either some e_p ($p \in \mathcal{A}_d, p \neq 1-d$) or to $\sum_{p \in \mathcal{A}_d, p \neq 1-d} e_p$. By Theorem 2.10 for $(d-1)$, there exists $g \in \mathfrak{G}_{d-1}$ such that $g.e_{d-1} = v'$ (we shift by 1 the indices and consider e_{d-1} and v' as vectors in $\mathbb{Z}^{\mathcal{A}_{d-1}}$). Let B be the matrix in $\mathfrak{K}_d$ associated to g by (3.1) and Proposition 3.2. The vector $B.e_{d-1}$ (now in $\mathbb{Z}^{\mathcal{A}_d}$) is equal to v because ${}^t h^\star.(v - B.e_{d-1}) = 0$. As $\mathfrak{K}_d \subset \mathfrak{G}_d$, this proves that v belongs to the orbit of e_{d-1} under $\mathfrak{G}_d$.
- In the general case, Lemma 3.4 says that one can find, in the subgroup generated L_{1-d}, L_{3-d}, a matrix B such that
$$B_{1-d,1-d} v_{1-d} + B_{1-d,3-d} v_{3-d} = 0.$$
Then, the $(1-d)$-component of $B.v$ is equal to zero and $B.v$ satisfies the hypothesis of the first case. We conclude that $B.v$, hence also v, belongs to the orbit of e_{d-1} under $\mathfrak{G}_d$.

This completes the proof of the proposition. □

Corollary 3.7. *Theorem* 2.11 *holds for d. In particular, the orbit of e_{1-d} under $\mathfrak{G}_d$ is equal to $\mathcal{O}_d$ from Proposition* 3.6.

Proof. Indeed, by Proposition 3.6, for any $p \in \mathcal{A}_d$, the orbit of e_{d-1} under $\mathfrak{G}_d$ contains $(-1)^{(d-1-p)/2} e_p$. □

We finally prove that $\mathfrak{G}_d$ is equal to $\mathfrak{G}'_d$ under the assumptions of this subsection.

Proof. Let $B \in \mathfrak{G}'_d$. From Corollary 3.7, there exists $B_0 \in \mathfrak{G}_d$ such that $B_0^{-1}.B$ fixes e_{1-d}. This means that $B_0^{-1}.B \in \mathfrak{K}_d \subset \mathfrak{G}_d$, hence B belongs to $\mathfrak{G}_d$. □

3.4. **The induction step in the even case.** In this section, we assume that $d \geqslant 4$ is even and that Theorems 2.9, 2.11 hold for $d-1$.

Proposition 3.8. *The stabilizer of e_{1-d} in $\mathfrak{G}_d$ is equal to $\mathfrak{K}_d$. It is generated by the L_p, $p \in \mathcal{A}_d, p \neq 1-d$.*

Proof. As $\mathfrak{G}_d \subset \mathfrak{G}'_d$, this stabilizer is contained in $\mathfrak{K}_d$. Let $\mathfrak{K}_d^\flat$ be the subgroup of $\mathfrak{K}_d$ generated by the L_p, $p \in \mathcal{A}_d, p \neq 1-d$. From the induction hypothesis $\mathfrak{G}_{d-1} = \mathfrak{G}'_{d-1}$ and Proposition 3.1, we deduce that the image of $\mathfrak{K}_d^\flat$ under φ_d is equal to $\mathfrak{G}'_{d-1}$. In order to conclude that $\mathfrak{K}_d^\flat$ is equal to $\mathfrak{K}_d$, it is sufficient to show, in view of Proposition 3.3, that the matrix

$$\begin{pmatrix} 1 & 2\,{}^t h^\star \\ 0 & \mathbf{1}_{d-1} \end{pmatrix} \tag{3.4}$$

belongs to $\mathfrak{K}_d^\flat$.

For $p \in \mathcal{A}_d$, $1-d < p < d-1$ define $M_p := L_{p+2}^{-1} \circ L_p \circ L_{p+2}$. This element of $\mathfrak{S}_{p,p+2}$ satisfies

$$M_p(e_p) = 2e_p + e_{p+2}, \quad M_p(e_{p+2}) = -e_p, \quad M_p(e_q) = e_q \quad \text{if } q \neq p, p+2.$$

Let

$$M := M_{d-3} \circ M_{d-5} \circ \ldots \circ M_{3-d}.$$

The matrix M belongs to $\mathfrak{K}_d^\flat$ and one has

$$\begin{aligned} M(e_{3-d}) &= e_{d-1} + 2 \sum_{p \in \mathcal{A}_d, 1-d<p<d-1} e_p, \\ M(e_p) &= -e_{p-2} \qquad \text{for } p \in \mathcal{A}_d, p > 3-d. \end{aligned}$$

Therefore $N := L_{d-1}^2 \circ M$ satisfies

$$N(e_{3-d}) = e_{d-1} - 2e_{1-d}, \quad N(e_p) = -e_{p-2} \quad \text{for } p \in \mathcal{A}_d, p > 3-d.$$

It follows that

$$N^{d-1}(e_p) = e_p + 2(-1)^{\frac{p-d+1}{2}} e_{1-d}, \qquad \forall p \in \mathcal{A}_d, p > 1-d.$$

Thus, the inverse of the matrix $N^{d-1} \in \mathfrak{K}_d^\flat$ has the required form (3.4). $\square$

Proposition 3.9. *The orbit of e_{d-1} under $\mathfrak{G}_d$ is equal to the set of primitive vectors in $\mathbb{Z}^{\mathcal{A}_d}$ which are congruent mod.2 to a vector in $\mathcal{E} = \{\sum_{p\in\mathcal{A}_d} \bar{e}_p\} \cup \{\bar{e}_p\}_{p\in\mathcal{A}_d}$.*

Proof. It is clear that vectors in the orbit of e_{d-1} under $\mathfrak{G}_d$ are primitive and congruent mod.2 to a vector in $\mathcal{E}$.

Conversely, let $v = \sum_{p\in\mathcal{A}_d} v_p e_p$ be a primitive vector in $\mathbb{Z}^{\mathcal{A}_d}$ which is congruent mod.2 to a vector in $\mathcal{E}$.

Lemma 3.10. *If $\sum_{p\in\mathcal{A}_d, p>1-d} (-1)^{\frac{d-1-p}{2}} v_p = 1$, then v belongs to the orbit of e_{d-1} under $\mathfrak{G}_d$.*

Proof. Let $v' := \sum_{p\in\mathcal{A}_d, p>1-d} v_p e_p$. From the hypothesis of the lemma, v' is a primitive vector, in particular it is not even. Therefore it is congruent mod.2 to one of the e_p (with $p \in \mathcal{A}_d$, $p > 1-d$) or to $\sum_{p\in\mathcal{A}_d, p>1-d} e_p$. From Theorem 2.11 for $d-1$, there exists $g \in \mathfrak{G}_{d-1}$ such that, after shifting the indices by 1, v' is the last column of g. By Propositions 3.1 and 3.3, there exists $B \in \mathfrak{K}_d$ such that v is the last column of B (one only needs even multiples of ${}^t h^\star$ because of the congruence condition). As $\mathfrak{K}_d$ is contained in $\mathfrak{G}_d$ by Proposition 3.8, we get the assertion of the lemma. $\square$

Lemma 3.11. *If $v_{1-d} = 1$, then v belongs to the orbit of e_{d-1} under $\mathfrak{G}_d$.*

Proof. For $w = \sum_{p\in\mathcal{A}_d} w_p e_p$, define $\phi(w) := \sum_{p\in\mathcal{A}_d, p>1-d} (-1)^{\frac{d-1-p}{2}} w_p$. Observe that, for $w \in \mathbb{Z}^{\mathcal{A}_d}$, $n \in \mathbb{Z}$, one has

$$\phi(L_{1-d}^n(w)) = \phi(w) + n w_{1-d}.$$

If $v_{1-d} = 1$, there exists $n \in \mathbb{Z}$ such that $L_{1-d}^n(v)$ satisfies the hypothesis of Lemma 3.10. Then $L_{1-d}^n(v)$ belongs to the orbit of e_{d-1} under $\mathfrak{G}_d$, and the same is true for v. $\square$

Lemma 3.12. *There exists $g \in \mathfrak{G}_d$ such that the first component of $g.v$ is equal to one.*

Proof. The argument is by infinite descent. It is clear that there exists $g \in \mathfrak{G}_d$ such that the first component of $g.v$ is positive. Then we may assume (replacing v by an appropriate $g.v$) that $v_{1-d} > 0$ and that, for any $g \in \mathfrak{G}_d$, the first component of $g.v$ is either $\leqslant 0$ or $\geqslant v_{1-d}$. We have to show that $v_{1-d} = 1$. We assume by contradiction that $v_{1-d} > 1$.

As v is primitive, there exists $p \in \mathcal{A}_d$, $p > 1-d$, such that v_p is not a multiple of v_{1-d}. Let $\bar{v}_{1-d} > 0$ be the smallest common divisor of v_p, v_{1-d}. One has $1 \leqslant \bar{v}_{1-d} < v_{1-d}$. By Lemma 3.4, there exists an element g in the subgroup $\mathfrak{S}_{1,p}$ generated by L_1, L_p such that the first coordinate of $g.v$ is equal to $\bar{v}_{1-d}$. This gives the required contradiction. □

The desired proposition follows from Lemmas 3.11 and 3.12. □

Similarly to the previous subsection, the induction step for d even follows from Propositions 3.8 and 3.9.

At this point, the inductive proofs of Theorems 2.9, 2.10, 2.11 are now complete.

4. Dehn twists and hyperelliptic Rauzy diagrams

In this section, we give an alternative proof of the precise version of Theorem 1.1 stated as Theorem 2.9 above. For this sake, we start with a general discussion of Dehn twists arising naturally from certain loops in Rauzy diagrams and then we specialize this discussion to the case of hyperelliptic Rauzy diagrams.

4.1. **General remarks on Rauzy diagrams and Dehn twists.**

Once again, we assume some familiarity with the reference [13] during this entire subsection.

Let $\mathcal{A}$ be an alphabet with $d \geqslant 2$ letters, let $\mathcal{R}$ be an arbitrary Rauzy class on $\mathcal{A}$, and let $\mathcal{D}$ be the associated Rauzy diagram.

4.1.1. *The surfaces M_π and their decorations.*

A partial reference for what follows is [13, Section 9.2].

For every $\pi \in \mathcal{R}$, we construct a *canonical* translation surface M_π with combinatorial data π whose length data λ^{can} and suspension data τ^{can} are given by

$$\lambda^{can}_\alpha = 1, \tau^{can}_\alpha = \pi_b(\alpha) - \pi_t(\alpha), \quad \forall \alpha \in \mathcal{A}.$$

Denote by g the genus of M_π and by s the cardinality of Σ_π. Recall that both g and s depend only on $\mathcal{R}$ and $d = 2g + s - 1$.

The surface is obtained by identifying parallel sides of a polygon P_π whose leftmost vertex, denoted by U_0 or V_0, is at $0 \in \mathbb{C}$. The rightmost vertex, denoted by U_d or V_d, is at d. The vertices above the real axis are denoted (from left to right) by $U_1, \dots, U_{d-1}$. The vertices below the real axis are denoted (from left to right) by $V_1, \dots, V_{d-1}$. As $\sum \tau_\alpha = 0$, Veech's zippered rectangle construction is not needed here.

We denote by Σ_π the set of marked points of M_π, we equip M_π with a basepoint $*_\pi = 1/2 \in \mathbb{C}$ and we set $O_\pi = d/2 \in \mathbb{C}$. We denote by $\mathcal{T} \not\ni *_\pi$ a curvilinear triangle whose sides are a curvilinear "vertical" segment $\eta = [U_{d-1}, V_{d-1}]$ through O_π and the sides $[U_{d-1}, U_d]$, $[V_{d-1}, V_d]$ of P_π.

We denote by Σ^*_π the subset of M_π consisting of O_π and the midpoints of the sides of P_π. Its cardinality is equal to $d+1$.

For each $\alpha \in \mathcal{A}$, we define an oriented loop θ_α in $M_\pi \setminus \Sigma_\pi$, based at $*_\pi$:

- We choose a simple path θ^t_α (resp. θ^b_α) from $*_\pi$ to the middle point of the top (resp. bottom) α-side of P_π passing through O_π via the horizontal segment $[*_\pi, O_\pi]$; this path is contained in the interior of P_π except for its endpoint.

- We ask that the $\theta_\alpha^\varepsilon$, $\alpha \in \mathcal{A}$, $\varepsilon \in \{t, b\}$ are disjoint except from their endpoints and $[*_\pi, O_\pi]$, and also disjoint from $[U_{d-1}, V_{d-1}]$ except at O_π.
- θ_α is the concatenation of θ_α^t and $(\theta_\alpha^b)^{-1}$ (so that θ_α is oriented upwards).

The difference $M_\pi \setminus \cup_{\alpha\in\mathcal{A}} \theta_\alpha$ is a finite union of open disks. Each of this disks contains exactly one point of Σ_π.

Recall that the fundamental group $\pi_1(M_\pi \setminus \Sigma_\pi, *_\pi)$ is a free group on $d = 2g + s - 1$ generators, namely, the classes of the θ_α, $\alpha \in \mathcal{A}$: see [13, Subsection 4.5], for instance.

4.1.2. *The homeomorphisms H_γ.* Consider an arrow $\gamma : \pi \to \pi'$ of $\mathcal{D}$. We claim that one can naturally associated to the arrow γ an orientation-preserving homeomorphism $H_\gamma : (M_\pi, \Sigma_\pi \cup \Sigma_\pi^* \cup \{*_\pi\}) \to (M_{\pi'}, \Sigma_{\pi'} \cup \Sigma_{\pi'}^* \cup \{*_{\pi'}\})$ which is uniquely defined modulo isotopy (amongst homeomorphisms sending $\Sigma_\pi \cup \Sigma_\pi^* \cup \{*_\pi\}$ to $\Sigma_{\pi'} \cup \Sigma_{\pi'}^* \cup \{*_{\pi'}\}$).

The homeomorphism H_γ is constructed as follows. We denote by α_t, α_b the letters of $\mathcal{A}$ such that $\pi_t(\alpha_t) = \pi_b(\alpha_b) = d$ and we let $B_\gamma \in SL(\mathbb{Z}^{\mathcal{A}})$ be the matrix associated to γ by the Rauzy–Veech algorithm / KZ-cocycle. We assume that γ is of top type (as the bottom case is completely similar).

- We cut the triangle $\mathcal{T}$ from P_π along η and glue[4] it again, after the appropriate translation, through the identification of the bottom α_t-side of P_π and the side $[U_{d-1}, U_d]$ of $\mathcal{T}$. We obtain in this way a polygon $P_{\pi'}^0$ with a pair of curvilinear "vertical" sides. This cutting and glueing process corresponds to the basic step of the Rauzy–Veech algorithm. The sides of $P_{\pi'}^0$ are labelled by $\mathcal{A}$ from 0 in the same cyclical order than for $P_{\pi'}$. In particular, the curvilinear "vertical" sides are labelled by α_t. The surface $M_{\pi'}^0$ obtained from $P_{\pi'}^0$ by glueing bottom and top sides of the same name is *canonically isomorphic* to M_π.
- We choose an orientation-preserving homeomorphism h from $P_{\pi'}^0$ onto $P_{\pi'}$ with the following properties
  - For each $\alpha \in \mathcal{A}$, h sends the top α side of $P_{\pi'}^0$ onto the top α side of $P_{\pi'}$, and the bottom α side of $P_{\pi'}^0$ onto the bottom α side of $P_{\pi'}$. This is done in a way which is compatible with the identification of top and bottom sides in $P_{\pi'}^0$ and $P_{\pi'}$.
  - For each $\alpha \in \mathcal{A}$, $\alpha \neq \alpha_t$, h sends the midpoint of the top (resp.bottom) α-side of $P_{\pi'}^0$ to the midpoint of the top (resp. bottom) α-side of $P_{\pi'}$.
  - h sends the point O_π (on the top α_t-side of $P_{\pi'}^0$) to the midpoint of the top α_t-side of $P_{\pi'}$.
  - h sends the midpoint of the bottom α_t-side of P_π (which lies inside $P_{\pi'}^0$) to $O_{\pi'}$ and h sends $*_\pi$ to $*_{\pi'}$.
- Finally, $H_\gamma : M_\pi \equiv M_{\pi'}^0 \to M_{\pi'}$ is the homeomorphism induced by h.

The reader can check that the homotopy class of H_γ (mod $\Sigma_\pi \cup \Sigma_\pi^* \cup \{*_\pi\}$) does not depend on the choices of h.

4.1.3. *Naming the marked points.* Let $\pi \in \mathcal{R}$. For each marked point a in Σ_π, define $\mathcal{A}(\pi, a) \subset \mathcal{A}$ as the subset of letters $\alpha \in \mathcal{A}$ such that a is the left endpoint of the α-sides of P_π. We have a partition

$$\mathcal{A} = \bigsqcup_{a\in\Sigma_\pi} \mathcal{A}(\pi, a). \tag{4.1}$$

[4]If γ were of bottom type, we would glue $\mathcal{T}$ to the top α_b-side of P_π.

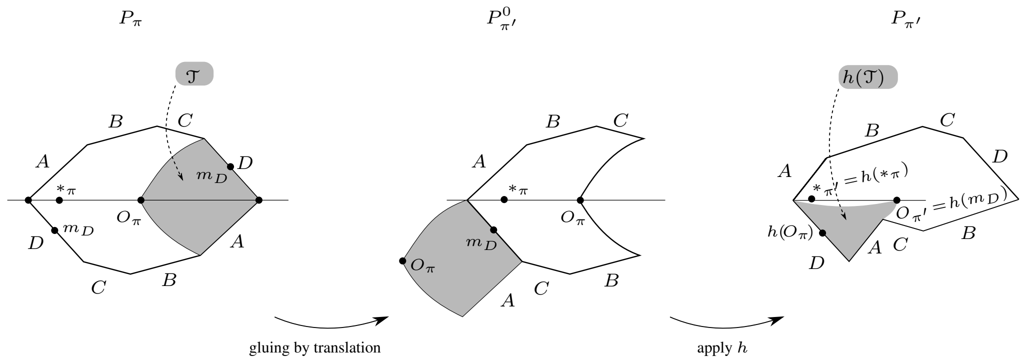

FIGURE 2. Construction of the homeomorphism H_γ.

It is easy to check that, for any arrow $\gamma : \pi \to \pi'$ of $\mathcal{D}$, the homeomorphism H_γ constructed above satisfies[5], for any $a \in \Sigma_\pi$

$$\mathcal{A}(\pi, a) = \mathcal{A}(\pi', H_\gamma(a)).$$

In other terms, the partition (4.1) above depends only on $\mathcal{R}$, not on π. Therefore, we can use (4.1) to name in a consistent way the points of the various Σ_π, $\pi \in \mathcal{R}$. The homeomorphisms H_γ respect the naming.

Remark 4.1. On the other hand, when γ is a loop, (i.e $\pi = \pi'$), the homeomorphism H_γ permutes in a non trivial way the points of Σ^*_π.

4.1.4. *The groupoid* $\Gamma(\mathcal{D})$. Consider the non-oriented Rauzy diagram $\widetilde{\mathcal{D}}$ associated to $\mathcal{D}$: it has the same vertices than $\mathcal{D}$; for each arrow $\gamma : \pi \to \pi'$ of $\mathcal{D}$, there are two arrows $\gamma^+ : \pi \to \pi'$ and $\gamma^- : \pi' \to \pi$ in $\widetilde{\mathcal{D}}$.

We define $\Gamma(\mathcal{D})$ as the groupoid of reduced oriented paths in $\widetilde{\mathcal{D}}$ (i.e., the groupoid of oriented paths quotiented by the cancellation rules $\gamma^+ \star \gamma^- = \gamma^- \star \gamma^+ = 1$).

To each arrow $\gamma^+ : \pi \to \pi'$ of positive type of $\widetilde{\mathcal{D}}$, we have constructed above a isotopy class $[H_\gamma]$ from $(M_\pi, \Sigma_\pi \cup \Sigma^*_\pi)$ to $(M_{\pi'}, \Sigma_{\pi'} \cup \Sigma^*_{\pi'})$ rel. $\Sigma_\pi \cup \Sigma^*_\pi$ which respects the naming of the points of Σ_π, $\Sigma_{\pi'}$. To an arrow γ^- of negative type, we associate the isotopy class of H_γ^{-1}. Compare with [13, Section 9.2].

We also define a groupoid $\mathrm{Mod}(\mathcal{R})$ in the following way. Its vertices are the elements of $\mathcal{R}$. The set $\mathrm{Mod}(\pi, \pi')$ of arrows from a vertex π to a vertex π' consists of the isotopy classes of orientation-preserving homeomorphisms from $(M_\pi, \Sigma_\pi \cup \Sigma^*_\pi \cup \{*_\pi\})$ to $(M_{\pi'}, \Sigma_{\pi'} \cup \Sigma^*_{\pi'} \cup \{*_{\pi'}\})$ rel. $\Sigma_\pi \cup \Sigma^*_\pi \cup \{*_\pi\}$ which respects the naming of the points of Σ_π, $\Sigma_{\pi'}$. In particular, the image of $\mathrm{Mod}(\pi) := \mathrm{Mod}(\pi, \pi)$ under the "forget $\Sigma^*_\pi \cup \{*_\pi\}$" homomorphism is the *pure* mapping class group of (M_π, Σ_π).

We extend the map $\gamma^+ \mapsto [H_\gamma]$, $\gamma^- \mapsto [H_\gamma^{-1}]$ to a morphism of groupoids from $\Gamma(\mathcal{D})$ to $\mathrm{Mod}(\mathcal{R})$. In particular, for each $\pi \in \mathcal{R}$, we have a group homomorphism from the fundamental group $\pi_1(\widetilde{\mathcal{D}}, \pi)$ to the pure modular group $\mathrm{Mod}(\pi)$ of (M_π, Σ_π).

Question 4.2. What is the image of this homomorphism from $\pi_1(\widetilde{\mathcal{D}}, \pi)$ to $\mathrm{Mod}(\pi)$?

Remark 4.3. We give an answer to this question for hyperelliptic Rauzy diagrams in Subsection 4.2 below.

[5]This is somewhat related to [13, Proposition 7.7].

4.1.5. *Action of H_γ on the fundamental groups.* Let $\gamma : \pi \to \pi'$ be an arrow of $\mathcal{D}$. We compute the homomorphism $\pi_1(\gamma) : \pi_1(M_\pi \setminus \Sigma_\pi, *_\pi) \to \pi_1(M_{\pi'} \setminus \Sigma_{\pi'}, *_{\pi'})$ induced by H_γ. We denote by α_w the winner of γ, by α_ℓ the loser of γ. Recall the generators θ_α, $\alpha \in \mathcal{A}$ of $\pi_1(M_\pi \setminus \Sigma_\pi, *_\pi)$. The corresponding generators for $\pi_1(M_{\pi'} \setminus \Sigma_{\pi'}, *_{\pi'})$ are denoted by θ'_α, $\alpha \in \mathcal{A}$. A direct inspection of our construction shows that:

Proposition 4.4. *One has $\pi_1(\gamma)(\theta_\alpha) = \theta'_\alpha$, for $\alpha \neq \alpha_\ell$, and*

$$\pi_1(\gamma)(\theta_{\alpha_\ell}) = \begin{cases} \theta'_{\alpha_\ell} \star (\theta'_{\alpha_w})^{-1} & \text{if } \gamma \text{ is of top type} \\ \\ (\theta'_{\alpha_w})^{-1} \star \theta'_{\alpha_\ell} & \text{if } \gamma \text{ is of bottom type} \end{cases}$$

4.1.6. *Pure cycles in $\mathcal{D}$.* A simple *oriented* loop in $\mathcal{D}$ is called a *pure cycle* if all its arrows have the same type (bottom or top). Equivalently, all its arrows have the same winner.

Let $\pi \in \mathcal{R}$. There are exactly two pure cycles through π. One is made of arrows of top type, with winner α_t. Its length is $d - \pi_b(\alpha_t)$. The other is made of arrows of bottom type, with winner α_b. Its length is $d - \pi_t(\alpha_b)$.

In the next proposition, Dehn twists in M_π along the curves θ_α, $\alpha \in \mathcal{A}$ are considered as elements of $\mathrm{Mod}(\pi)$ by choosing a representative which is supported in a neighborhood of θ_α and *exchanges* O_π and the midpoint of the α-sides of P_π (these two points are the only points of Σ^*_π lying on θ_α).

Proposition 4.5. *Let $\pi \in \mathcal{R}$ and let Γ be a pure cycle through π. If Γ is of top type, the image of $\Gamma \in \pi_1(\widetilde{\mathcal{D}}, \pi)$ in $\mathrm{Mod}(\pi)$ is the left Dehn twist along θ_{α_t}. If Γ is of bottom type, the image of Γ in $\mathrm{Mod}(\pi)$ is the right Dehn twist along θ_{α_b}.*

Proof. This fact can be deduced by direct inspection on M_π or by computing the action on fundamental groups (in a similar way to Proposition 4.4 above). □

4.2. **Hyperelliptic Rauzy diagrams, Dehn twists and braid groups.** In this subsection, we restrict ourselves to hyperelliptic Rauzy diagrams. Let $d \geqslant 2$ be an integer and consider the hyperelliptic Rauzy class $\mathcal{R}_d$ equipped of its central vertex $\pi^* = \pi^*(d)$, and the hyperelliptic Rauzy diagram $\mathcal{D}_d$ introduced in Subsection 2.1.

4.2.1. *Elementary simple loops in $\mathcal{D}_d$.* The canonical surface M_{π^*} is hyperelliptic thanks to the hyperelliptic involution τ_{π^*} given by central symmetry at the point $O_{\pi^*} = d/2$. In particular, $O_{\pi^*} = d/2$ and the midpoints of the sides of the polygon P_{π^*} contains Weierstrass points of M_{π^*}. Moreover, the marked point of M_{π^*} is a Weierstrass point when d is even, while the marked points of M_{π^*} are exchanged by the hyperelliptic involution.

In this setting, Proposition 4.5 says that the image in $\mathrm{Mod}(\pi^*)$ of a non-oriented loop $\gamma' \in \pi_1(\widetilde{\mathcal{D}_d}, \pi^*)$ associated to an elementary simple loop in $\mathcal{R}_d$ is a Dehn twist exchanging O_{π^*} with the midpoint of a side of P_{π^*}. In other words, we have computed the action of γ' at the homotopical level (as promised in Remark 2.2): again, it depends only on the type and winner of γ, and the homotopical action of a loop of bottom type is the inverse of the homotopical action of a loop of top type and the same winner.

Remark 4.6. The fact that elementary simple loops in $\mathcal{R}_d$ act by Dehn twists implies that one can not expect the hyperelliptic Rauzy–Veech groups $\mathfrak{G}_d$ to coincide with the full symplectic group in Theorem 1.1. More precisely, Dehn twists act on homology by symplectic transvections, so that $\mathfrak{G}_d$ is generated by $d = 2g$ symplectic transvections when d is even. However, it is known that one can not generate $\mathrm{Sp}(2g, \mathbb{Z})$ with fewer than $2g+1$ symplectic transvections (cf. [6, Proposition 6.5]).

4.2.2. *Symmetric mapping class groups and braid groups.* The Dehn twists associated to the elementary simple loops in $\mathcal{R}_d$ commute with the hyperelliptic involution τ_{π^*} of M_{π^*}. Therefore, the image of the homomorphism from $\pi_1(\widetilde{\mathcal{D}}_d, \pi^*)$ to $\mathrm{Mod}(\pi^*)$ is contained in the *symmetric mapping class subgroup*, that is, the centralizer of τ_{π^*} in $\mathrm{Mod}(\pi^*)$.

It follows that the Dehn twists associated to elementary simple loops in $\mathcal{R}_d$ are lifts to M_{π^*} of certain elements of a *braid group*[6]. More precisely, the hyperelliptic translation surface M_{π^*} can be thought of as the hyperelliptic Riemann surface $y^2 = (x-b_1)\dots(x-b_{d+1})$ equipped with the Abelian differential dx/y (whose zeroes are at the points at infinity) for an appropriate choice of configuration $\{b_1, \dots, b_{d+1}\}$ of pairwise distinct points in $\mathbb{C}$. Here, the subset $\{b_1, \dots, b_{d+1}\}$ of Weierstrass points correspond to the set of $\Sigma^*_{\pi^*}$, and, for the sake of concreteness, we make our choices so that b_{d+1} corresponds to O_{π^*} while b_n, $1 \leqslant n \leqslant d$ correspond to the midpoints of sides of P_{π^*}. In this context, we see that the Dehn twists associated to elementary simple loops in $\mathcal{R}_d$ are lifts to M_{π^*} of the elements θ_n, $1 \leqslant n \leqslant d$, of the braid group B_{d+1} exchanging b_n and b_{d+1}.

Note that $\{\theta_n : 1 \leqslant n \leqslant d\}$ is not a system of Artin[7] standard generators $\{\sigma_j : 1 \leqslant j \leqslant d\}$ where σ_j exchanges b_j and b_{j+1}, but it is not hard to see that σ_j can be written in terms of θ_j and θ_{j+1} (by conjugation). In particular, $\{\theta_n : 1 \leqslant n \leqslant d\}$ generates the braid group B_{d+1} and, *a fortiori*, the image of the homomorphism from $\pi_1(\widetilde{\mathcal{D}}_d, \pi^*)$ to $\mathrm{Mod}(\pi^*)$ is precisely the symmetric mapping class group $\mathrm{SMod}(\pi^*) \simeq B_{d+1}$.

Remark 4.7. The symmetric mapping class group $\mathrm{SMod}(\pi^*)$ is an infinite-index subgroup of $\mathrm{Mod}(\pi^*)$ corresponding to the orbifold fundamental group[8] of projectivized hyperelliptic connected components of the moduli spaces of translation surfaces: see, e.g., Looijenga–Mondello [10]. Thus, we have just shown that, in a certain sense, the hyperelliptic Rauzy diagrams "see" the topology of the projectivized hyperelliptic connected components of the moduli spaces of translation surfaces.

4.2.3. *Monodromy representations of braid groups.* The elements of $\mathrm{SMod}(\pi^*)$ act on the homology M_{π^*}. This induces a natural monodromy representation

$$\rho_{d+1} : B_{d+1} \to \mathrm{Sp}(H_1(M_{\pi^*}, \mathbb{Z}))$$

of the braid group B_{d+1}.

It follows from our discussion above of the homomorphism from $\pi_1(\widetilde{\mathcal{D}}_d, \pi^*)$ to $\mathrm{Mod}(\pi^*)$ that the hyperelliptic Rauzy-Veech group $\mathfrak{G}_d$ coincides with the image $\rho_{d+1}(B_{d+1})$ of the monodromy representation ρ_{d+1}.

As it turns out, the image of ρ_{d+1} was described by A'Campo [1, Théorème 1]:

Theorem 4.8 (A'Campo)**.** *Let $d \geqslant 2$. The image of ρ_{d+1} contains the congruence subgroup of level two of Sp$(H_1(M_{\pi^*}, \mathbb{Z}))$. Moreover, the reduction of $\rho_{d+1}(B_{d+1})$ mod.2 is isomorphic to a symmetric group of order $d+1$, resp. three, for $d \neq 3$, resp. $d = 3$.*

Remark 4.9. Notice that Theorem 4.8 only describes the action on on the absolute homology, while Corollary 2.8 describes the action on the full relative homology. For this reason, we get slightly different groups in the special case $d = 3$.

In this way, we recover the description of the hyperelliptic Rauzy-Veech group $\mathfrak{G}_d = \rho_{d+1}(B_{d+1})$ in Theorems 1.1 and 2.9.

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[6]The braid group B_m is the fundamental group of the space $\mathbb{C}^{\langle m \rangle}$ of configurations of finite subsets of $\mathbb{C}$ of cardinality m based at an arbitrarily fixed configuration $* \in \mathbb{C}^{\langle m \rangle}$.

[7]See [6, Section 9.2], for instance.

[8]An interesting consequence of this fact is the non-connectedness of the hyperelliptic Teichmüller spaces.

APPENDIX A. A PINCHING AND TWISTING GROUP WITH SMALL ZARISKI CLOSURE

Let ρ be the third symmetric power of the standard representation of $SL(2,\mathbb{R})$. In concrete terms, ρ is constructed as follows. Consider the basis $\mathcal{B}=\{X^3, X^2Y, XY^2, Y^3\}$ of the space V of homogenous polynomials of degree 3 on two variables X and Y. By letting $g=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL(2,\mathbb{R})$ act on X and Y as $g(X)=aX+cY$ and $g(Y)=bX+dY$, we obtain an induced action $\rho(g)$ on V whose matrix in the basis $\mathcal{B}$ is

$$\rho\begin{pmatrix} a & b \\ c & d\end{pmatrix}=\begin{pmatrix} a^3 & a^2b & ab^2 & b^3 \\ 3a^2c & a^2d+2abc & b^2c+2abd & 3b^2d \\ 3ac^2 & bc^2+2acd & ad^2+2bcd & 3bd^2 \\ c^3 & c^2d & cd^2 & d^3\end{pmatrix}$$

Note that the faithful representation ρ is the unique irreducible four-dimensional representation of $SL(2,\mathbb{R})$. Furthermore, the matrices $\rho(g)$ preserve the symplectic structure on V associated to the matrix

$$J=\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1/3 & 0 \\ 0 & -1/3 & 0 & 0 \\ 1 & 0 & 0 & 0\end{pmatrix}$$

Indeed, a direct calculation shows that if $g=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$, then

$$^t\rho(g)\cdot J\cdot\rho(g)=\begin{pmatrix} 0 & 0 & 0 & -(ad-bc)^3 \\ 0 & 0 & \frac{(ad-bc)^3}{3} & 0 \\ 0 & -\frac{(ad-bc)^3}{3} & 0 & 0 \\ (ad-bc)^3 & 0 & 0 & 0\end{pmatrix}$$

where $^t\rho(g)$ stands for the transpose of $\rho(g)$.

Denote by $\mathcal{M}$ the group generated by the matrices

$$A=\rho\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix}=\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1\end{pmatrix}\quad\text{and}\quad B=\rho\begin{pmatrix} 1 & 0 \\ 1 & 1\end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1\end{pmatrix}$$

On one hand, the group $\mathcal{M}$ has small Zariski closure.

Proposition A.1. *The group $\mathcal{M}$ is not Zariski dense in $Sp(V)$.*

Proof. Note that ρ is a *polynomial*[9] homomorphism from $SL(2,\mathbb{R})$ to $Sp(V)$. In particular, it follows that the image $H=\rho(SL(2,\mathbb{R}))$ of ρ is Zariski closed in $Sp(V)$: see, e.g., Corollary 4.6.5 in Witte-Morris book [12]. Since $SL(2,\mathbb{Z})$ is a Zariski dense subgroup of $SL(2,\mathbb{R})$ generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1\end{pmatrix}$, we see that the Zariski closure of the group $\mathcal{M}$ is $H=\rho(SL(2,\mathbb{R}))$ ($\simeq SL(2,\mathbb{R})$ because ρ is faithful). This proves the proposition (since H is a proper linear algebraic subgroup of $Sp(V)$). $\square$

On the other hand, the group $\mathcal{M}$ is pinching and twisting in the sense of Avila–Viana (see [3] and [9, Section 2]):

[9] That is, the entries of $\rho(g)\in Sp(V)$ depend polynomially on the entires of $g\in SL(2,\mathbb{R})$.

Proposition A.2. *The matrix $A.B \in \mathcal{M}$ is a pinching element*[10] *and the matrix $A \in \mathcal{M}$ is twisting*[11] *with respect to the pinching element $A.B \in \mathcal{M}$.*

Proof. The first assertion follows from the fact that

$$9+4\sqrt{5} > \frac{3+\sqrt{5}}{2} > \frac{3-\sqrt{5}}{2} > \frac{1}{9+4\sqrt{5}}$$

are the eigenvalues of $A.B \in \mathcal{M}$.

The second assertion is established by the following reasoning. The columns of

$$M = \begin{pmatrix} -\frac{1}{4}+\frac{(9+4\sqrt{5})}{4} & 1-\frac{(3+\sqrt{5})}{2} & 1-\frac{(3-\sqrt{5})}{2} & -\frac{1}{4}+\frac{(9-4\sqrt{5})}{4} \\ \frac{9}{8}+\frac{3(9+4\sqrt{5})}{8} & -2+\frac{(3+\sqrt{5})}{2} & -2+\frac{(3-\sqrt{5})}{2} & \frac{9}{8}+\frac{3(9-4\sqrt{5})}{8} \\ -\frac{15}{8}+\frac{3(9+4\sqrt{5})}{8} & \frac{(3+\sqrt{5})}{2} & \frac{(3-\sqrt{5})}{2} & -\frac{15}{8}+\frac{3(9-4\sqrt{5})}{8} \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

consist of eigenvectors of $A.B$. Thus, $T = M^{-1}\cdot A\cdot M$ is the matrix of A in the corresponding basis of eigenvectors of $A.B$. By definition, A is twisting with respect to $A.B$ when all entries of T and all of its 2×2 minors associated to Lagrangian planes are non-zero. As it turns out, this last property holds because a direct computation reveals that T and the matrix $T^{\wedge 2}$ of 2×2 minors are given by:

$$T = \begin{pmatrix} \frac{8(5+2\sqrt{5})}{25} & \frac{2(5+3\sqrt{5})}{25} & \frac{(5+\sqrt{5})}{25} & \frac{1}{(5\sqrt{5})} \\ -\frac{6(5+3\sqrt{5})}{25} & \frac{2(5+\sqrt{5})}{25} & \frac{7}{5\sqrt{5}} & -\frac{3(-5+\sqrt{5})}{25} \\ \frac{3(5+\sqrt{5})}{25} & -\frac{7}{5\sqrt{5}} & -\frac{2(-5+\sqrt{5})}{25} & \frac{6(-5+3\sqrt{5})}{25} \\ -\frac{1}{5\sqrt{5}} & \frac{(5-\sqrt{5})}{25} & \frac{2}{5}-\frac{6}{5\sqrt{5}} & -\frac{(8(-5+2\sqrt{5})}{25} \end{pmatrix}$$

and

$$T^{\wedge 2} = \begin{pmatrix} \frac{56}{25}+\frac{24}{5\sqrt{5}} & \frac{32}{25}+\frac{16}{5\sqrt{5}} & \frac{18}{25}+\frac{6}{5\sqrt{5}} & \frac{6}{25}+\frac{2}{5\sqrt{5}} & \frac{2}{25}+\frac{2}{5\sqrt{5}} & \frac{1}{25} \\ -\frac{32}{25}-\frac{16}{5\sqrt{5}} & \frac{6}{25}+\frac{2}{5\sqrt{5}} & \frac{9}{25}+\frac{9}{5\sqrt{5}} & \frac{3}{25}+\frac{3}{5\sqrt{5}} & \frac{11}{25} & -\frac{2}{25}+\frac{2}{5\sqrt{5}} \\ \frac{6}{25}+\frac{2}{5\sqrt{5}} & -\frac{3}{25}-\frac{3}{5\sqrt{5}} & \frac{13}{25} & -\frac{4}{25} & -\frac{3}{25}+\frac{3}{5\sqrt{5}} & \frac{6}{25}-\frac{2}{5\sqrt{5}} \\ \frac{18}{25}+\frac{6}{5\sqrt{5}} & -\frac{9}{25}-\frac{9}{5\sqrt{5}} & -\frac{36}{25} & \frac{13}{25} & -\frac{9}{25}+\frac{9}{5\sqrt{5}} & \frac{18}{25}-\frac{6}{5\sqrt{5}} \\ -\frac{2}{25}-\frac{2}{5\sqrt{5}} & \frac{11}{25} & \frac{9}{25}-\frac{9}{5\sqrt{5}} & \frac{3}{25}-\frac{3}{5\sqrt{5}} & \frac{6}{25}-\frac{2}{5\sqrt{5}} & -\frac{32}{25}+\frac{16}{5\sqrt{5}} \\ \frac{1}{25} & \frac{2}{25}-\frac{2}{5\sqrt{5}} & \frac{18}{25}-\frac{6}{5\sqrt{5}} & \frac{6}{25}-\frac{2}{5\sqrt{5}} & \frac{32}{25}-\frac{16}{5\sqrt{5}} & \frac{56}{25}-\frac{24}{5\sqrt{5}} \end{pmatrix}$$

$\square$

In summary, these propositions say that $\mathcal{M}$ is the desired group: it is pinching and twisting, but not Zariski dense in $Sp(V)$.

Remark A.3. Observe that the group $\rho(SL(2,\mathbb{Z}))$ does not contain Galois-pinching[12] elements of $Sp(V)$ in the sense of [9] because $H = \rho(SL(2,\mathbb{R}))$ has rank 1. Alternatively, this fact can be shown as follows. A straightforward computation reveals that the characteristic polynomial of $\rho(g)$ is

$$(x^2 - \mathrm{tr}(g)\det(g)x + \det(g)^3)\cdot(x^2 - \mathrm{tr}(g)(\mathrm{tr}(g)^2 - 3\det(g))x + \det(g)^3)$$

---

[10] Its eigenvalues are all real with distinct moduli.

[11] $A(F)\cap F' = \{0\}$ for all $A.B$-invariant isotropic subspaces $F\subset V$ and all $A.B$-invariant coisotropic subspaces $F'\subset V$ with $\dim(F)+\dim(F') = 4$.

[12] Pinching elements whose characteristic polynomials have the largest possible Galois group among reciprocal integral polynomials (namely, hyperoctahedral groups).

and, consequently, the eigenvalues of $\rho(g)$ are

$$\frac{1}{2}\det(g)\left(\mathrm{tr}(g) \pm \sqrt{\mathrm{tr}(g)^2 - 4\det(g)}\right),$$

and

$$\frac{1}{2}\left(\mathrm{tr}(g)(\mathrm{tr}(g)^2 - 3\det(g)) \pm (\mathrm{tr}(g)^2 - \det(g))\sqrt{\mathrm{tr}(g)^2 - 4\det(g)}\right).$$

Therefore, the Galois group of the characteristic polynomial $\rho(g)$, $g \in SL(2, \mathbb{Z})$, is not the largest possible among reciprocal polynomials of degree four.

Remark A.4. It seems *unlikely* to find pinching and twisting monoids of symplectic matrices which are not Zariski dense in the context of the Kontsevich–Zorich cocycle. Indeed, Filip's classification theorem [7] says that, modulo finite-index and compact factors, a Kontsevich–Zorich monodromy $\mathcal{M}_V$ has Zariski closure $\mathrm{Sp}(V)$, $\mathrm{SU}(p,q)$, $\mathrm{SO}^*(2n)$, $\wedge^k\mathrm{SU}(p,1)$ or some spin groups. Thus, *all* matrices in $\mathcal{M}_V$ have two eigenvalues with the same modulus *unless* the Zariski closure of $\mathcal{M}_V$ is isomorphic to $\mathrm{Sp}(V)$ modulo finite-index and compact factors. It follows that if $\mathcal{M}_V$ is pinching, then $\mathcal{M}_V$ is Zariski dense in $\mathrm{Sp}(V)$ modulo finite-index and compact factors.

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